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ON A FAMILY OF LAGUERRE METHODS TO FIND MULTIPLE ROOTS OF NONLINEAR EQUATIONS

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Abstract. There are several methods for solving a nonlinear algebraic equation having roots of a given multiplicity m. Here we compare a family of Laguerre methods of order three as well as two others of the same order and show that Euler-Cauchy's method is best. We discuss the conjugacy maps and the effect of the extraneous roots on the basins of attraction.

Keywords: Iterative methods; Order of convergence; Rational maps; Basin of attraction; Julia sets; Conjugacy classes.
Mathematics Subject Classification: 65H05, 65B99

1. Introduction

There is a vast literature on the solution of nonlinear equations and nonlinear systems, see for example Ostrowski [1], Traub [2], Neta [3] and the recent book by Petković et al [4] and references therein. Most of the algorithms are for finding a simple root of a nonlinear equation $f(x) = 0$, i.e. for a root α we have $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. In this paper we are interested in the case that α is a root of multiplicity $m > 1$. There are very few methods for multiple roots when the multiplicity is known. The first one is due to Schröder [5] and it is also referred to as modified Newton,

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \tag{1}$$

The method is based on Newton's method for the function $G(x) = \sqrt[m]{f(x)}$ which obviously has a simple root at α, the multiple root with multiplicity m of $f(x)$.

Another method based on the same G is Laguerre's method

$$x_{n+1} = x_n - \frac{\lambda\frac{f(x_n)}{f'(x_n)}}{1 + sgn(\lambda - m)\sqrt{\left(\frac{\lambda-m}{m}\right)\left[(\lambda - 1) - \lambda\frac{f(x_n)f''(x_n)}{f'(x_n)^2}\right]}} \tag{2}$$

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where λ ($\neq 0$, m) is a real parameter. When $f(x)$ is a polynomial of degree n, this method with $\lambda = n$ is the ordinary Laguerre method for multiple roots, see Bodewig [6]. This method converges cubically. Some special cases are:

- Euler-Cauchy for $\lambda = 2m$

$$x_{n+1} = x_n - \frac{2m\frac{f(x_n)}{f'(x_n)}}{1 + \sqrt{(2m-1) - 2m\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}. \tag{3}$$

- Halley for $\lambda \to 0$ after rationalization

$$x_{n+1} = x_n - \frac{\frac{f(x_n)}{f'(x_n)}}{\frac{m+1}{2m} - \frac{f(x_n)f''(x_n)}{2f'(x_n)^2}}. \tag{4}$$

- Ostrowski for $\lambda \to \infty$

$$x_{n+1} = x_n - \frac{\sqrt{m}\frac{f(x_n)}{f'(x_n)}}{\sqrt{1 - \frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}. \tag{5}$$

- Hansen-Patrick family [7] for $\lambda = m(1/\nu + 1)$

$$x_{n+1} = x_n - \frac{m(\nu+1)\frac{f(x_n)}{f'(x_n)}}{\nu + \sqrt{\big(m(\nu+1) - \nu\big) - m(\nu+1)\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}. \tag{6}$$

Petković et al [8] have shown the equivalence between Laguerre family (2) and Hansen-Patrick family (6). When $\lambda \to m$ the method becomes second order given by (1). Two other cubically convergent methods that sometimes mistaken as members of Laguerre's family are: Euler-Chebyshev [2] given by

$$x_{n+1} = x_n - \left(\frac{m(3-m)}{2} + \frac{m^2}{2}\frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}, \tag{7}$$

and Osada's method [9] given by

$$x_{n+1} = x_n - \frac{1}{2}m(m+1)\frac{f(x_n)}{f'(x_n)} + \frac{(m-1)^2}{2}\frac{f'(x_n)}{f''(x_n)}. \tag{8}$$

Other variations on Chebyshev's method are given by [10].

Osada [11] has shown that the error for Laguerre's method (2) is given by

$$e_{n+1} = K_3(m,\lambda)\, e_n^3 + O(e_n^4), \tag{9}$$

where the asymptotic error constant, $K_3(m,\lambda)$ is given by

$$K_3(m,\lambda) = A_1(m,\lambda)\left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right)^2 - A_2(m)\,\frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}, \tag{10}$$

with

$$A_1(m,\lambda) = \frac{1}{2m(m+1)^2}\left(1 - \frac{1}{\lambda - m}\right),$$

$$A_2(m) = \frac{1}{m(m+1)(m+2)}.$$

For Euler-Cauchy, the asymptotic error constant is

$$K_3(m,2m) = \frac{m-1}{2m^2(m+1)^2}\left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)}\right)^2 - A_2(m)\,\frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \tag{11}$$

For Halley's method ($\lambda \to 0$) the asymptotic error constant is

$$K_3(m, \lambda \to 0) = \frac{1}{2m^2(m+1)} \left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)} \right)^2 - A_2(m) \frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \tag{12}$$

For Ostrowski's method ($\lambda \to \infty$) the asymptotic error constant is given by [1]

$$K_3(m, \lambda \to \infty) = \frac{1}{2m(m+1)^2} \left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)} \right)^2 - A_2(m) \frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \tag{13}$$

The asymptotic error constant for Euler-Chebyshev's method (see Traub [2]) is

$$K_3 = \frac{m+3}{2m^2(m+1)^2} \left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)} \right)^2 - \frac{1}{m(m+1)(m+2)} \frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \tag{14}$$

The asymptotic error constant for Osada's method [9] is

$$K_3 = \frac{(m+1)^2}{2m^2(m-1)} \left(\frac{f^{(m+1)}(\alpha)}{f^{(m)}(\alpha)} \right)^2 - \frac{1}{m} \frac{f^{(m+2)}(\alpha)}{f^{(m)}(\alpha)}. \tag{15}$$

If we define the efficiency index of a method of order p as

$$I = p^{1/d}, \tag{16}$$

where d is the number of function- (and derivative-) evaluation per step then all these methods have the same efficiency of $3^{1/3} = 1.442$. There is no indication which method is superior by looking at the error constants. In the next sections we will discuss basins of attraction and conjugacy maps for the polynomial $((z-a)(z-b))^m$ which is the generalization of a quadratic polynomial to the case of multiple roots.

2. Corresponding Conjugacy Maps for Quadratic polynomials

Given two maps f and g from the Riemann sphere into itself, an analytic conjugacy between the two maps is a diffemorphism h from the Riemann sphere onto itself such that $h \circ f = g \circ h$. Here we consider only quadratic polynomials raised to m^{th} power.

Theorem 2.1. *(Euler-Cauchy's method (3)) For a rational map $R_p(z)$ arising from Euler-Cauchy's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ to*

$$S(z) = z(z-1)\left[1 + sgn(z^2-1)\right].$$

Proof. Let $p(z) = ((z-a)(z-b))^m$, $a \neq b$ and let M be the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \dfrac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p \left(\frac{ub-a}{u-1} \right) = u(u-1)\left[1 + sgn(u^2-1)\right].$$

□

Theorem 2.2. *(Halley's method (4)) For a rational map $R_p(z)$ arising from Halley's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ to*

$$S(z) = z^3.$$

Proof. Let $p(z) = ((z-a)(z-b))^m,\ a \neq b$ and let M be the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \dfrac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p \left(\frac{ub-a}{u-1} \right) = u^3.$$

□

Theorem 2.3. *(Ostrowski's method (5)) For a rational map $R_p(z)$ arising from Ostrowski's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ to*

$$S(z) = \frac{z\left[sgn(z+1)\sqrt{z^2+1}-1\right]}{sgn(z+1)\sqrt{z^2+1}-z}.$$

Proof. Let $p(z) = ((z-a)(z-b))^m,\ a \neq b$ and let M be the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \dfrac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p \left(\frac{ub-a}{u-1} \right) = \frac{u\left[sgn(u+1)\sqrt{u^2+1}-1\right]}{sgn(u+1)\sqrt{u^2+1}-u}.$$

□

Theorem 2.4. *(Euler-Chebyshev's method (7)) For a rational map $R_p(z)$ arising from Euler-Chebyshev's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ to*

$$S(z) = z^3 \frac{z+2}{2z+1}.$$

Proof. Let $p(z) = ((z-a)(z-b))^m,\ a \neq b$ and let M be the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \dfrac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p \left(\frac{ub-a}{u-1} \right) = u^3 \frac{u+2}{2u+1}.$$

□

Theorem 2.5. *(Osada's method (8)) For a rational map $R_p(z)$ arising from Osada's method applied to $p(z) = ((z-a)(z-b))^m$, $a \neq b$, $R_p(z)$ is conjugate via the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ to*

$$S(z) = z^3 \frac{(m-1)z + 2m}{2mz + m - 1}.$$

Proof. Let $p(z) = ((z-a)(z-b))^m$, $a \neq b$ and let M be the Möbius transformation given by $M(z) = \dfrac{z-a}{z-b}$ with its inverse $M^{-1}(u) = \dfrac{ub-a}{u-1}$, which may be considered as a map from $\mathbb{C} \cup \{\infty\}$. We then have

$$S(u) = M \circ R_p \circ M^{-1}(u) = M \circ R_p \left(\frac{ub-a}{u-1} \right) = u^3 \frac{(m-1)u + 2m}{2mu + m - 1}.$$

□

In the next 2 sections, we will analyze the basins of attraction to compare all these third order methods for multiple roots. The idea of using basins of attraction was initiated by Stewart [12] and followed by the works of Amat et al [13], [14], [15], and [16], Scott et al [18] and Chun et al [17]. The only paper comparing basins of attraction for methods to obtain multiple roots is due to Neta et al [19]. They have not considered some of the methods appearing here.

3. Extraneous fixed points

In solving a nonlinear equation iteratively we are looking for fixed points which are zeros of the given nonlinear function. Many iterative methods have fixed points that are not zeros of the function of interest. Those points are called extraneous fixed points (see Vrcsay and Gilbert [20]). Those points could be attractive which will trap an iteration sequence and give erroneous results. Even if those extraneous fixed points are repulsive or indifferent they can complicate the situation by converging to a root not close to the initial guess.

All of the methods discussed here can be written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} H_f(x_n).$$

Clearly the root α of $f(x)$ is a fixed point of the method. The points $\xi \neq \alpha$ at which $H_f(\xi) = 0$ are also fixed points of the family, since the second term on the right vanishes.

It is easy to see that $H_f(x_n)$ for our methods is given in Table 3.

From the Table one can see that H_f does not vanish for Euler-Cauchy, Halley and Ostrowski's methods. Therefore there are no extraneous fixed points for these methods.

Theorem 3.1. *There are two extraneous fixed points for Euler-Chebyshev's method. They are the roots of*

$$\frac{f(\xi)f''(\xi)}{f'(\xi)^2} = \frac{m-3}{m}. \tag{17}$$

Method	H_f
Euler-Cauchy	$\dfrac{2m}{1+\sqrt{(2m-1)-2m\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}$
Halley	$\dfrac{1}{\frac{m+1}{2m}-\frac{f(x_n)f''(x_n)}{2f'(x_n)^2}}$
Ostrowski	$\dfrac{\sqrt{m}}{\sqrt{1-\frac{f(x_n)f''(x_n)}{f'(x_n)^2}}}$
Euler-Chebyshev	$\frac{m(3-m)}{2}+\frac{m^2}{2}\frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{f'(x_n)}$
Osada	$\frac{1}{2}m(m+1)-\frac{(m-1)^2}{2}\frac{f'(x_n)}{f''(x_n)}\frac{f'(x_n)}{f(x_n)}$

Proof. The extraneous fixed points can be found by solving (17). For the polynomial $(z^2-1)^m$ this leads to the equation

$$\frac{2mz^2-z^2-1}{2mz^2}=\frac{m-3}{m}$$

for which the roots are $\xi=\pm\frac{1}{\sqrt{5}}$.

These fixed points are attractive. Vrcsay and Gilbert [20] show that if the points are attractive then the method will give erroneous results. If the points are repulsive then the method may not converge to a root near the initial guess.

The poles are at $z=0$.

□

Theorem 3.2. *There are two extraneous fixed points for Osada's method. They are the roots of*

$$\frac{f(\xi)f''(\xi)}{f'(\xi)^2}=\frac{(m-1)^2}{m(m+1)}. \tag{18}$$

Proof. The extraneous fixed points can be found by solving (18). For the polynomial $(z^2-1)^m$ this leads to the equation

$$\frac{2mz^2-z^2-1}{2mz^2}=\frac{(m-1)^2}{m(m+1)},$$

for which the roots are $\xi=\pm\sqrt{\frac{m+1}{5m-3}}$.

These fixed points are repulsive for all $m>1$.

The poles are at $z=\pm\dfrac{1}{\sqrt{2m-1}}$. □

4. Numerical experiments

We have used the above methods for 6 different polynomials having multiple roots with multiplicity $m=2,3,4,5$.

In our first example, we have taken the polynomial

$$p_1(z)=(z^2-1)^2 \tag{19}$$

whose roots $z = \pm 1$ are both real and of multiplicity $m = 2$. The results are presented in Figures 1-5. Notice that the darker the shade in each basin, the faster the convergence to the root. Euler-Cauchy's method (Figure 1) for this example converged in 1 iteration to the closest root and in order to avoid having black points everywhere, we have used two different colors. This only happened for $(z^2-1)^m$. Halley's method (Figure 2) is slightly better than Ostrowski's (Figure 3). Euler-Chebyshev's (Figure 4) and Osada's method (Figure 5) are not as good. Notice that these two methods are the only ones with extraneous fixed points and poles along the real line.

Our next example is also having double roots. The polynomial have the three roots of unity,

$$p_2(z) = (z^3 - 1)^2. \tag{20}$$

The results are presented in Figures 6-10. Again Euler-Cauchy's (Figure 6) and Ostrowski's (Figure 8) methods performed better than Halley's method (Figure 2). The Euler-Chebyshev's method (Figure 9) was the worst and Osada's method (Figure 10) only slightly better than that.

The third example is a polynomial whose roots are all of multiplicity four. The roots are the three roots of unity, i.e.

$$p_3(z) = (z^3 - 1)^4. \tag{21}$$

The results are presenetd in Figures 11-15. Euler-Cauchy's method was the best followed by Ostrowski's method, Halley's method, Euler-Chebyshev's and Osada's schemes. The change in multiplicity, did not change the conclusions.

The fourth example is a polynomial whose roots are all of multiplicity three. The roots are $-2.68261500670705 \pm .358259359924043i, 1.36523001341410$, i.e.

$$p_4(z) = (z^3 + 4z^2 - 10)^3. \tag{22}$$

The results are presented in Figures 16-20. Based on these figure, we arrive at the same conclusions as before.

In our next example we took the polynomial

$$p_5(z) = (z^4 - 1)^5 \tag{23}$$

where the roots are symmetrically located on the axes. In some sense this is similar to the first example, since in both cases we have an even number of roots. The results are shown in Figures 21-25. Again we can see the best is Euler-Cauchy's method (Figure 21) and the worst is Osada's method (Figure 25).

In our last example we have the 5 roots of unity all with multiplicity three

$$p_6(z) = (z^5 - 1)^3. \tag{24}$$

The results are given in Figures 26-30. Again we can see the best is Euler-Cauchy's method (Figure 26) and the worst is Osada's method (Figure 30).

Conclusions

In all six examples, we find that the best is Euler-Cauchy's method and the worst are those with extraneous fixed points and poles on the real line, namely Euler-Chebyshev's and Osada's schemes. Notice that Neta et al [19] have found that Halley's method is one of the best, but now that we have compared it to Euler-Cauchy's method, we realized that the latter is even better.

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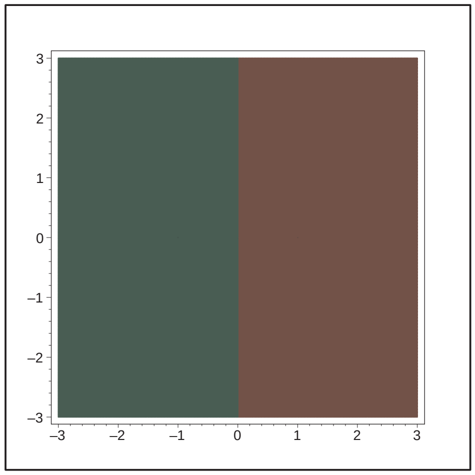

FIGURE 1. Euler-Cauchy's method for the roots of the polynomial $(z^2-1)^2$

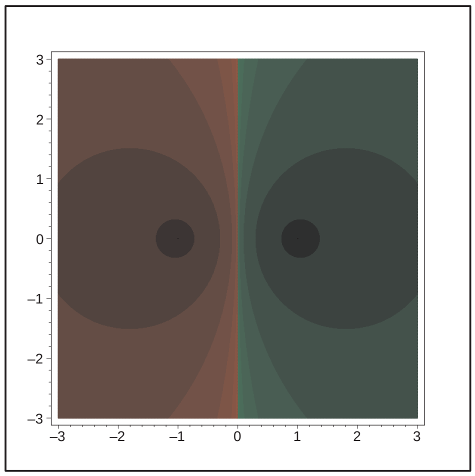

FIGURE 2. Halley's method for the roots of the polynomial $(z^2-1)^2$

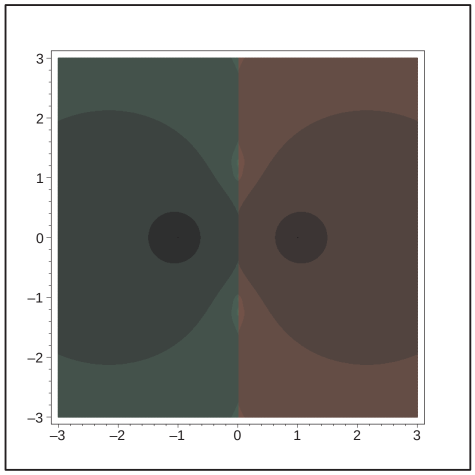

FIGURE 3. Ostrowski's method for the roots of the polynomial $(z^2-1)^2$

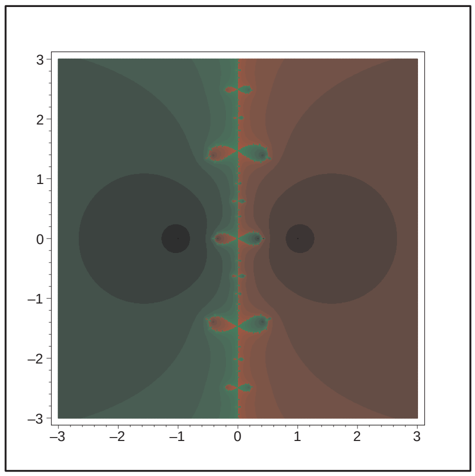

FIGURE 4. Euler-Chebyshev's method for the roots of the polynomial $(z^2 - 1)^2$

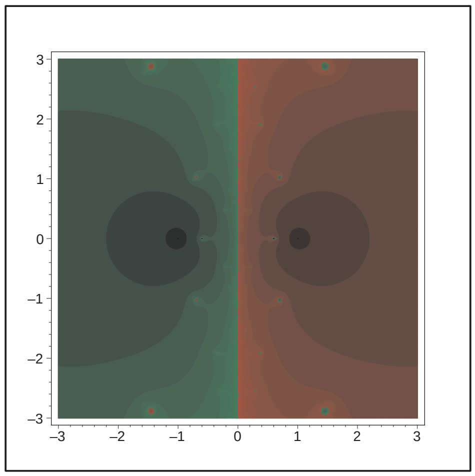

FIGURE 5. Osada's method for the roots of the polynomial $(z^2 - 1)^2$

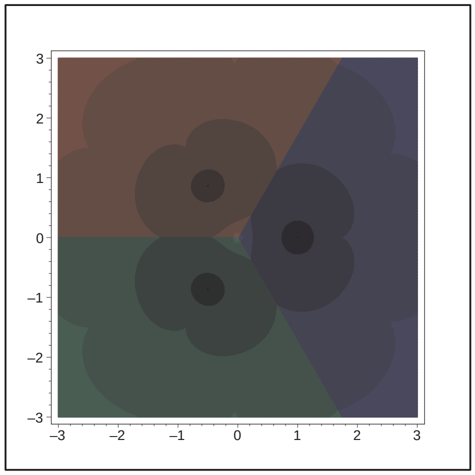

FIGURE 6. Euler-Cauchy's method for the roots of the polynomial $(z^3-1)^2$

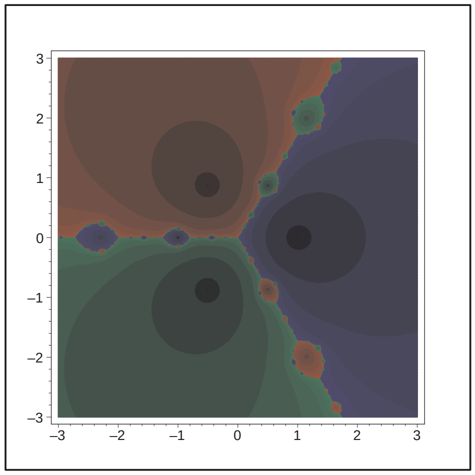

FIGURE 7. Halley's method for the roots of the polynomial $(z^3 - 1)^2$

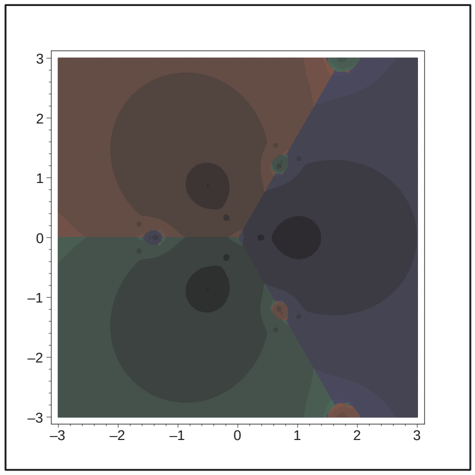

FIGURE 8. Ostrowski's method for the roots of the polynomial $(z^3 - 1)^2$

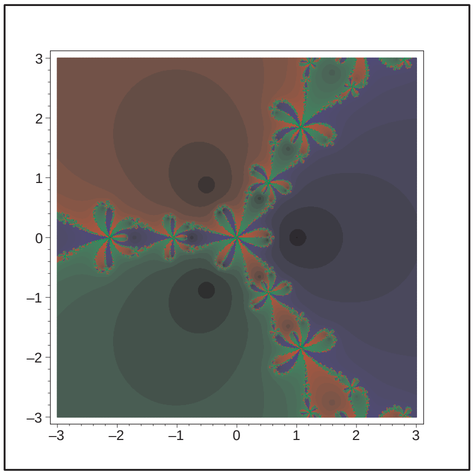

FIGURE 9. Euler-Chebyshev's method for the roots of the polynomial $(z^3 - 1)^2$

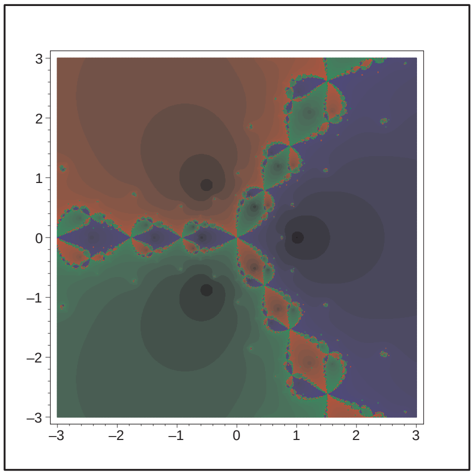

FIGURE 10. Osada's method for the roots of the polynomial $(z^3-1)^2$

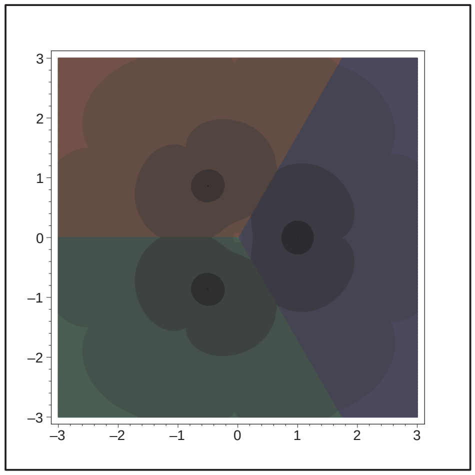

FIGURE 11. Euler-Cauchy's method for the roots of the polynomial $(z^3-1)^4$

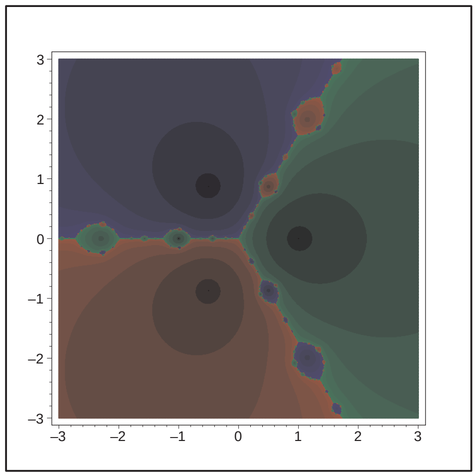

FIGURE 12. Halley's method for the roots of the polynomial $(z^3 - 1)^4$

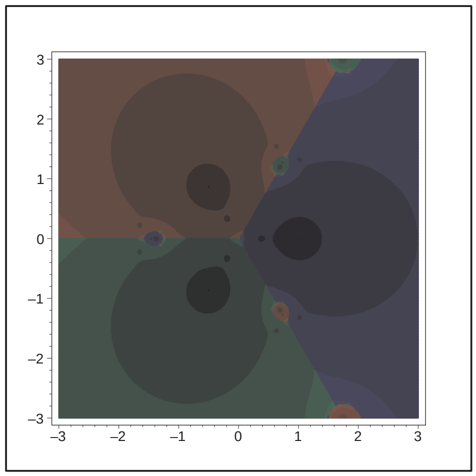

FIGURE 13. Ostrowski's method for the roots of the polynomial $(z^3-1)^4$

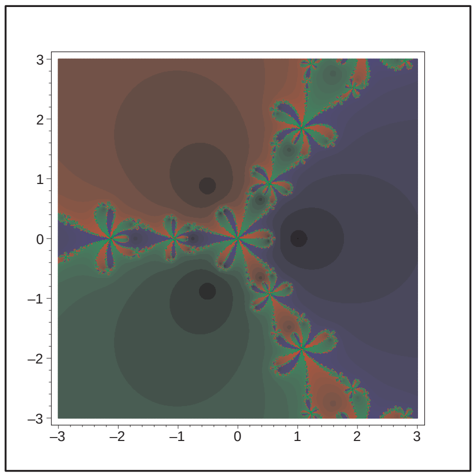

FIGURE 14. Euler-Chebyshev's method for the roots of the polynomial $(z^3 - 1)^4$

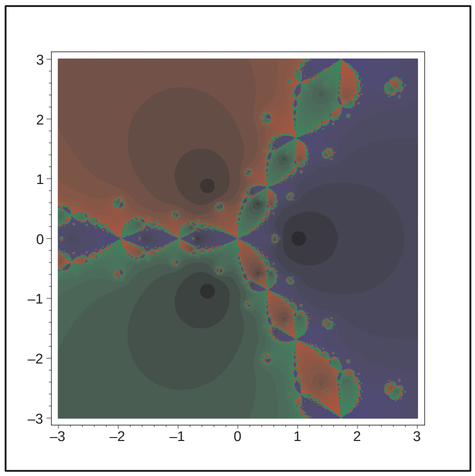

FIGURE 15. Osada's method for the roots of the polynomial $(z^3-1)^4$

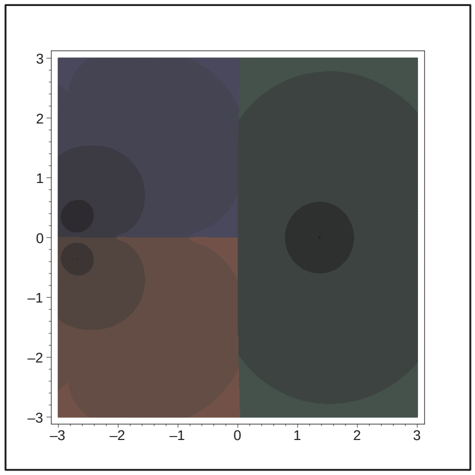

FIGURE 16. Euler-Cauchy's method for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$

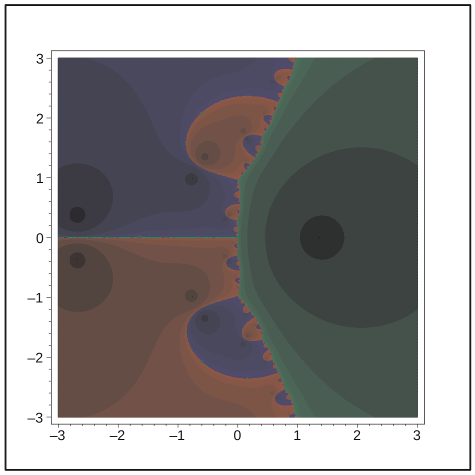

FIGURE 17. Halley's method for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$

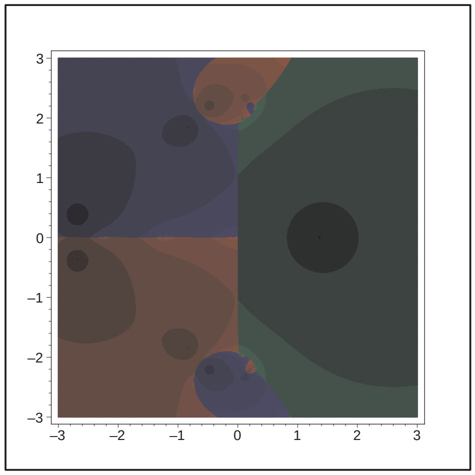

FIGURE 18. Ostrowski's method for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$

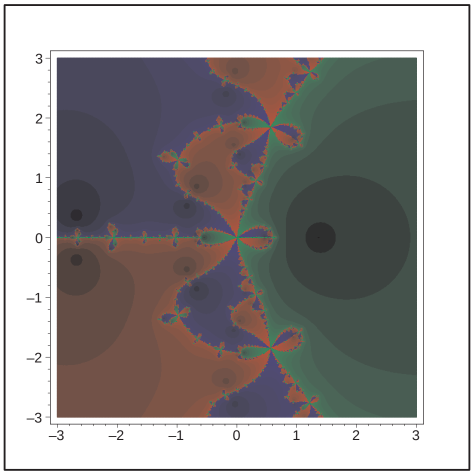

FIGURE 19. Euler-Chebyshev's method for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$

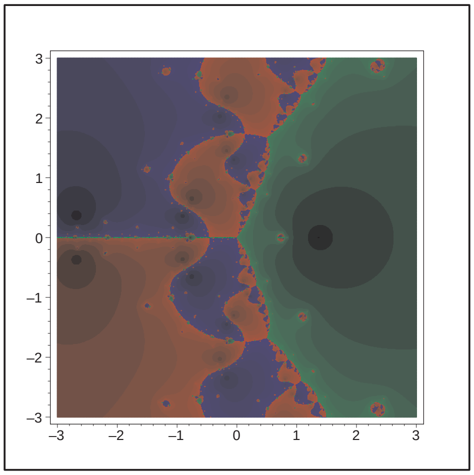

FIGURE 20. Osada's method for the roots of the polynomial $(z^3 + 4z^2 - 10)^3$

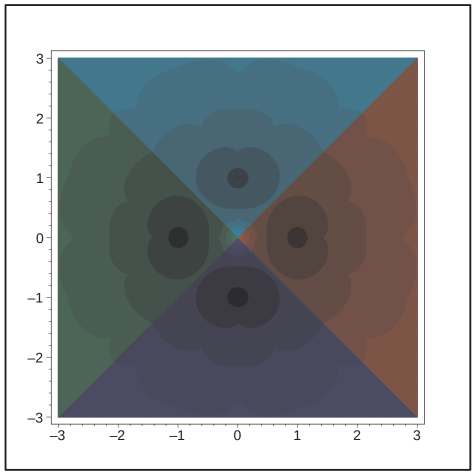

FIGURE 21. Euler-Cauchy's method for the roots of the polynomial $(z^4-1)^5$

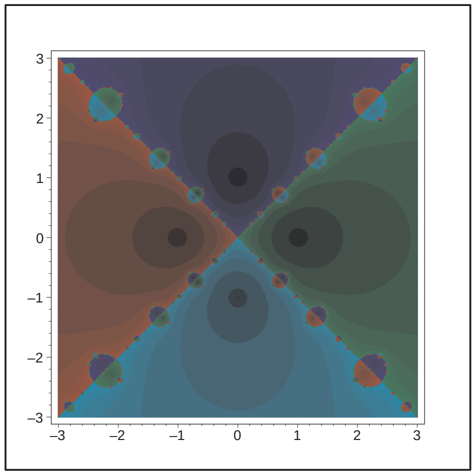

FIGURE 22. Halley's method for the roots of the polynomial $(z^4 - 1)^5$

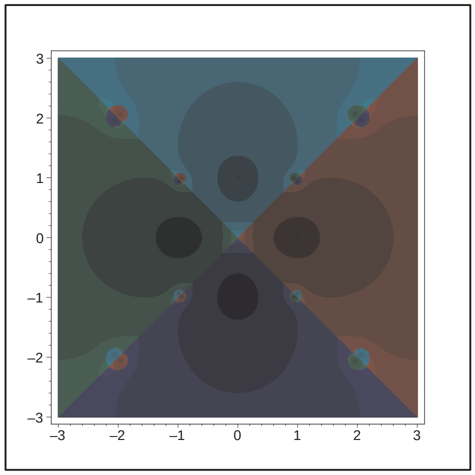

FIGURE 23. Ostrowski's method for the roots of the polynomial $(z^4 - 1)^5$

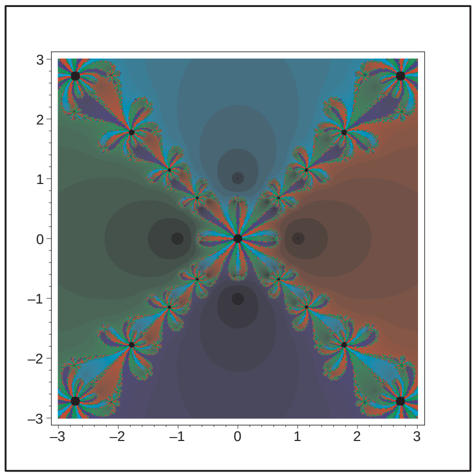

FIGURE 24. Euler-Chebyshev's method for the roots of the polynomial $(z^4 - 1)^5$

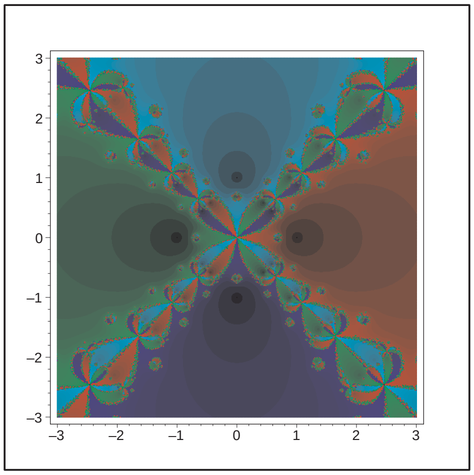

FIGURE 25. Osada's method for the roots of the polynomial $(z^4 - 1)^5$

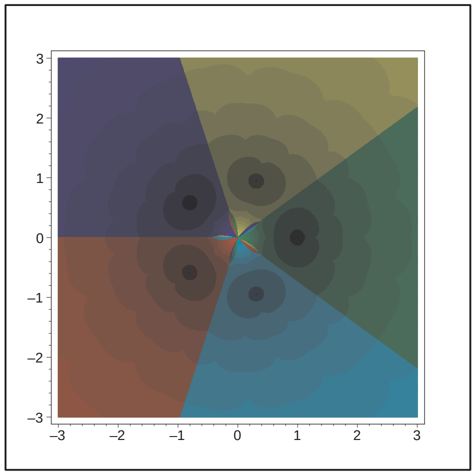

FIGURE 26. Euler-Cauchy's method for the roots of the polynomial $(z^5-1)^3$

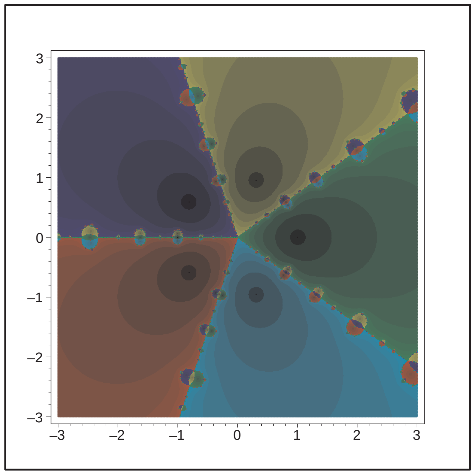

FIGURE 27. Halley's method for the roots of the polynomial $(z^5 - 1)^3$

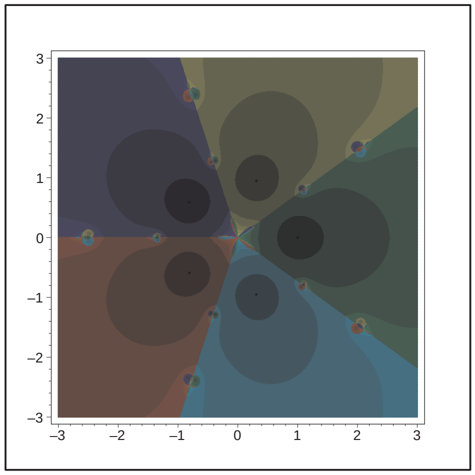

FIGURE 28. Ostrowski's method for the roots of the polynomial $(z^5-1)^3$

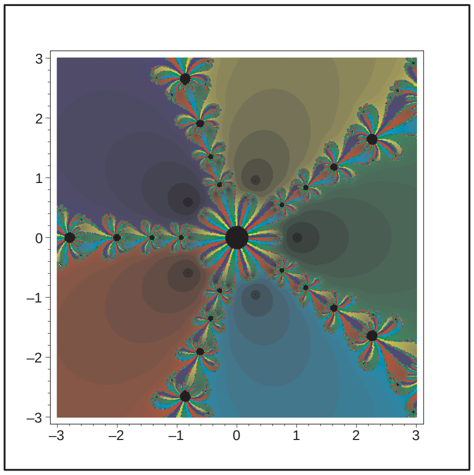

FIGURE 29. Euler-Chebyshev's method for the roots of the polynomial $(z^5 - 1)^3$

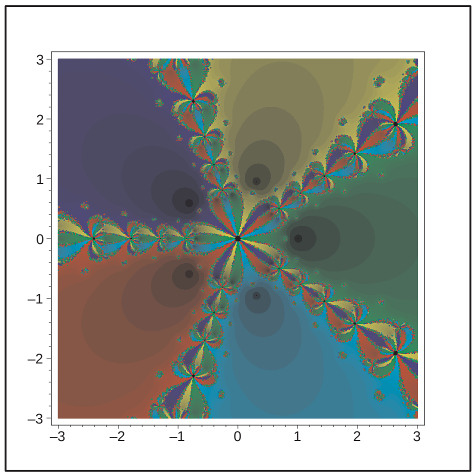

FIGURE 30. Osada's method for the roots of the polynomial $(z^5 - 1)^3$